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Best Approximation and Moduli of Smoothness for Doubling Weights

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In this paper we relate the rate of weighted polynomial approximation to some weighted moduli of smoothness for so-called doubling weights. We shall also consider the problem in a more restrictive sense for generalized Jacobi weights with zeros in the interval of approximation. These zeros constitute a special problem that has not been resolved so far in the literature.

1. INTRODUCTION

A central topic in polynomial approximation is to connect the rate of approximation to smoothness properties of functions. The core of this theory lies in Jackson's theorem and its Stechkin-type converses. Although for the trigonometric case the direct and inverse results had been proven to be matching pairs long time ago, for algebraic polynomial approximation the correct formulation of the Jackson inequality and its converse was done only fairly recently in [2]. To formulate the results we need the following definitions: Let w be a weight function on the interval $[-1, 1]$. The best

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weighted approximation with weight w of a function f by polynomials of degree at most n is defined as

$$E_n(f)_w = \inf_{\deg(P_n) \leqslant n} \|w(f - P_n)\|,$$

where, and everywhere in this paper, $\|\cdot\|$ denotes the supremum norm on $[-1, 1]$. Note that in this setting both w and f can have singularities. One of the main objectives of approximation theory is to characterize the rate of decrease of $E_n(f)$ in terms of some smoothness properties of functions. In most cases smoothness is defined via the so-called rth differences of the function,

$$\Delta_h^r g(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} g(x + (r/2 - s)\, h).$$

I turns out, that for a correct formulation of the Jackson theorem in the algebraic case we need to modify the step size h here by the function

$$\varphi(x) = \sqrt{1 - x^2};$$

therefore we set

$$\omega_\varphi^r(f, \tau)_w = \sup_{0 < h \leqslant \tau} \|w(x)\, \Delta_{h\varphi(x)}^r f(x)\|,$$

where it is understood that if any of the arguments $(x + (r/2 - s)\, h\varphi(x))$ in the expression of $\Delta_{h\varphi(x)}^r f(x)$ lies outside $[-1, 1]$, then we set this difference equal to 0. With this we have in the unweighted ($w \equiv 1$) case the two inequalities (see [2, Theorems 7.2.1 and 7.2.4])

$$E_n(f) \leqslant C\omega_\varphi^r\left(f, \frac{1}{n}\right) \tag{1.1}$$

and

$$\omega_\varphi^r\left(f, \frac{1}{n}\right) \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f). \tag{1.2}$$

In particular, if $0 < \alpha < r$, then

$$E_n(f) = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^r(f, t) = O(t^\alpha). \tag{1.3}$$

The aim of this paper is to extend these results to weighted cases. For some special classes of weights polynomial approximation was considered, e.g, in the works [1, 2, 5]. However, the case when the weight function has a zero inside the interval poses special difficulties, and has not been discussed in the literature.

The estimates we prove are in the form of (1.1) and (1.2), and they are true regardless if f has singularities or not. It is not our aim here to analyse when the right hand side in the analogue of (1.1) tends to zero, although we shall comment on this problem at the end of this section after formulating Theorem 1.4.

We start with a very general class of weights, namely the so called doubling weights. We say that the weight function w defined on $[-1, 1]$ is a doubling weight, if there is a constant L (called the doubling constant) such that

$$\int_{2I} w \leqslant L \int_{I} w \tag{1.4}$$

for all intervals $I \subset [-1, 1]$, where $2I$ denotes the interval that we obtain when we enlarge I twice from its midpoint (note that parts of $2I$ may lie outside $[-1, 1]$, where we set $w=0$). However, this is a too general weight concept for weighted approximation (e.g., a doubling weight need not be bounded and may vanish on a set of positive measure); therefore we shall consider certain averages of the weight over intervals of small length related to the degree of the approximation. We set

$$w_n(x) = \frac{1}{\Delta_n(x)} \int_{x-\Delta_n(x)}^{x+\Delta_n(x)} w(u)\, du, \tag{1.5}$$

where

$$\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$$

is the familiar function from polynomial approximation. For convenience we also define $w_0(x)$ to be $w_1(x)$. The doubling condition is equivalent (see [3, Lemma 7.1]) to

$$w_n(y) \leqslant K(1 + n\, |x-y| + n\, |\sqrt{1-x^2} - \sqrt{1-y^2}|)^s\, w_n(x), \tag{1.6}$$

for $n \in \mathrm{N}$ and $x, y \in [-1, 1]$ with some constants K and s.

Now the analogue of (1.1) and (1.2) is

THEOREM 1.1. *Let w be a doubling weight and let r be a positive integer. Then there is a constant C depending only on r and the doubling constant of w such that we have for any f*

$$E_n(f)_{w_n} \leqslant C \omega_\varphi^r \left(f, \frac{1}{n}\right)_{w_n}. \tag{1.7}$$

Conversely,

$$\omega_{\varphi}^{r+2}\left(f, \frac{1}{n}\right)_{w_n} \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{w_k}. \tag{1.8}$$

Note that the upper estimate can be applied with ω^{r+2}, as well, so (1.7) and (1.8) constitute matching positive and converse estimates, e.g., we can state

COROLLARY 1.2. *If w is a doubling weight, then for $0<\alpha<r$ we have*

$$E_n(f)_{w_n} = O(n^{-\alpha}) \Leftrightarrow \omega_{\varphi}^{r+2}(f, 1/n)_{w_n} = O(n^{-\alpha}). \tag{1.9}$$

Thus, as opposed to the unweighted case, here we need somewhat higher order of smoothness for the characterization of the order of approximation $E_n(f) = O(n^{-\alpha})$. This is necessary when we are dealing with such general weight concept as doubling weight, as the next example shows. Let $w(x) = |x|^{-\gamma}$ for some $0<\gamma<1$. Then this is a doubling weight, and it is easy to see that $w_n(x) \sim \min\{|x|^{-\gamma}, n^{\gamma}\}$, where $\sim$ means that the ratio of the two sides is bounded from below and from above by two positive constants. Consider the function $f(x) = x \log 1/|x|$. It is well known, that this is a smooth function in the sense that

$$\|\Delta_h^2 f\| \leqslant Ch, \qquad 0<h<1.$$

Therefore,

$$\|w_n \Delta_h^2 f\| \leqslant Cn^{\gamma}h, \qquad 0<h<1,$$

and hence $\omega_{\varphi}^2(f, 1/n)_{w_n} = O(n^{\gamma-1})$, which implies by (1.7)

$$E_n(f)_{w_n} = O(n^{\gamma-1}).$$

On the other hand,

$$\omega_{\varphi}^1(f, 1/n)_{w_n} \geqslant w_n(0)\, |f(1/2n) - f(-1/2)| \sim n^{\gamma-1} \log n,$$

which shows that (1.9) is not true with ω^{r+2} replaced by ω^r $(r=1)$ as in (1.3) (we do not know if the inequality (1.9) holds with ω^{r+1} instead of ω^{r+2}).

Next we discuss a class of weights where ω^{r+2} can be replaced by ω^r in (1.8). We say that w satisfies the A^* property if there is a constant L (called the A^* constant of w) such that for all intervals $I \subset [-1, 1]$ and $x \in I$ we have

$$w(x) \leqslant L \frac{1}{|I|} \int_I w. \tag{1.10}$$

This is stronger than the doubling property (see [3, Theorems 2.1 and 6.1]), nevertheless many bounded weights satisfy it. Let us also note that (1.10) allows high order zeros for w.

For A^* weight we have the following

THEOREM 1.3. *Let w be an A^* weight, and let r be a positive integer. Then there is a constant C depending only on r and the A^* constant of w such that for any f*

$$E_n(f)_{w_n} \leqslant C\omega_\varphi^r\left(f, \frac{1}{n}\right)_{w_n}, \tag{1.11}$$

and conversely

$$\omega_\varphi^r\left(f, \frac{1}{n}\right)_{w_n} \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{w_k}. \tag{1.12}$$

Let us mention that the proof yields for the converse inequality the somewhat sharper form

$$\omega_\varphi^r\left(f, \frac{1}{n}\right)_{w_n} \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{w_n}. \tag{1.13}$$

Indeed, this is a sharper form, for the properties of w imply that $w_n(x) \leqslant Cw_k(x)$ for all x and $k \leqslant n$ with a constant C. Notice also that if w is a doubling or an A^* weight, then so are w_n for every m with a doubling or A^* constant independent of m. If one combines this with the fact that in such cases we have $(w_m)_n \sim w_m$ for $n \geqslant m$, one can obtain various inequalities involving the averages w_m, for example, in (1.11) and (1.13) one can replace the weights w_n by any w_m with $m \leqslant n$.

Next we address the question if one can replace in our theorems the weight w_n by w, i.e., if we can get the complete analogues of (1.1) and (1.2) for the weighted case. To indicate the difficulty in such extensions, let us consider the following example. Let w be a so called generalized Jacobi weight of the form

$$w(x) = \prod_{j=1}^{N} |x - x_j|^{\gamma_j}, \tag{1.14}$$

where $-1 \leqslant x_1 < \cdots < x_N \leqslant 1$ are distinct points, and where the γ_j's are positive numbers. Consider now any zero of this weight, say x_j lying in $(-1, 1)$, fix $h > 0$, and for an $\eta > 0$ that will tend to zero consider the continuous function f_η that vanishes outside $[x_j - \eta, x_j + \eta]$, it equals $\eta^{-\gamma_j/2}$

at x_j and linear on the intervals $[x_j-\eta, x_j]$ and $[x_j, x_j+\eta]$. Clearly, for this f_η the norm $\|wf_\eta\|_{[-1,1]}$ tends to 0 as $\eta \to 0$. However, for $\eta < h$

$$w(x_j+rh/2)\,|\Delta_h^r f(x_j+rh/2)| = w(x_j+rh/2)\,|f_\eta(x_j)| \geqslant c\left(\frac{h}{\sqrt{\eta}}\right)^{\gamma_j},$$

which is as large as we wish if $\eta \to 0$.

This example shows the difficulty in forming the weighted moduli of smoothness when there is a zero in the weight, for, as we have just seen, in weighted spaces the norm is not bounded under translation. As another indication that internal zeros may cause trouble let us recall the following results from [4] on weighted Jackson–Favard inequalities.

THEOREM A. *Let w be a generalized Jacobi weight* (1.14) *and r a positive integer. Suppose, that if $x_j \neq \pm 1$, then, the corresponding γ_j is either not an integer, or it is bigger than r. Then there is a constant C depending only on r and w, such that for all functions f for which $f^{(r-1)}$ is locally absolutely continuous on $(-1, 1)\backslash\{x_j\}_{j=1}^N$, we have*

$$E_n(f)_w \leqslant \frac{C}{n^r}\,(\|wf^{(r)}\varphi^r\| + \|wf\|). \tag{1.15}$$

THEOREM B. *Let w be a generalized Jacobi weight* (1.14) *and r a positive integer. Suppose, that for some $x_j \neq \pm 1$ the corresponding γ_j is a positive integer not bigger than r. Then there exists a function f such that $f^{(r-1)}$ is locally absolutely continuous on $(-1, 1)\backslash\{x_j\}_{j=1}^N$,*

$$\|wf^{(r)}\varphi^r\| + \|wf\|$$

is finite, and still

$$\limsup_{n\to\infty} \frac{E_n(f)_w}{\log n/n^r} > 0. \tag{1.16}$$

Therefore, in the presence of internal zeros the Jackson–Favard inequality is not necessarily true.

To overcome the aforementioned difficulties with forming weighted moduli we shall modify their definition in the following way. Let w be the generalized Jacobi weight of (1.14). We assume $x_1 = -1$ and $x_N = 1$ with the agreement that γ_1 resp. γ_N is zero if -1 resp. 1 is not originally among the points x_j. For an $h > 0$ consider the intervals

$$I_{1,h} = [-1, -1+h^2], \qquad I_{N,h} = [1-h^2, 1],$$

and

$$I_{j,h} = [x_j - h, x_j + h], \qquad 1 < j < N, \tag{1.17}$$

and

$$J_{1,h} = [-1 + h^2, x_2 - h], \qquad J_{N-1,h} = [x_{N-1} + h, 1 - h^2],$$

and

$$J_{j,h} = [x_j + h, x_{j+1} - h], \qquad 1 < j < N - 1. \tag{1.18}$$

Then the $I_{j,h}$'s are lying around x_j, the $J_{j,h}$'s are the complementary intervals and altogether these intervals form a decomposition of $[-1, 1]$.

In forming our modified moduli of smoothness we shall consider the weighted norm of $\Delta^r_{h\varphi} f$ on the intervals $J_{j,h}$, and to their sum we add some additional terms that "connect" the parts of f lying on these intervals. We set

$$\omega^r_\varphi(f, \tau)^*_w = \sum_{j=1}^{N-1} \sup_{0 < h \leqslant \tau} \|w(x)\, \Delta^r_{h\varphi(x)} f(x)\|_{J_{j,h}} + \sum_{j=1}^{N} \inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f - P_{r-1})\|_{I_{j,\tau}}, \tag{1.19}$$

where we agree that when forming the norm

$$\|w(x)\, \Delta^r_{h\varphi(x)}\, f(x)\|_{J_{j,h}}$$

we set the symmetric difference equal to zero if the interval $[x - (r/2)\, h\varphi(x), x + (r/2)\, h\varphi(x)]$ does not belong to $J_{j,h}$.

As one can see this modulus of smoothness consists of two parts: one part is formed like the usual moduli of smoothness (in [2] such moduli were called main part moduli), and the other part is just the best approximation of f by polynomials of the fixed degree $r-1$ over small intervals. To see why we need the second part consider that if f is linear on each of $[x_j, x_{j+1}]$, then the first part vanishes identically (provided $r > 1$), so without this second part our modulus of smoothness could not possibly be used in Jackson-type estimates. We also note that similar second parts have already been utilized in approximation theory, see, e.g., [2, Chap. 11].

Now with this new modulus of smoothness we can prove

THEOREM 1.4. *Let w be a generalized Jacobi weight* (1.14). *Then there is a constant C depending only on r and the weight w such that for any f*

$$E_n(f)_w \leqslant C\omega_\varphi^r\left(f, \frac{1}{n}\right)_w^*, \tag{1.20}$$

and

$$\omega_\varphi^r\left(f, \frac{1}{n}\right)_w^* \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_w. \tag{1.21}$$

Note that the first part of this theorem improves upon Theorem A. Let us also note that (1.20) is true regardless of smoothness properties of f. However, it may happen that the right hand side in (1.20) does not tend to zero as $n \to \infty$. In fact, *for generalized Jacobi weights* (1.14) *the modulus of smoothness $\omega_\varphi^r(f, t)_w^*$ tends to zero as $t \to 0$ if and only if the function f is continuous on $[-1, 1]$ except perhaps at the points x_j, where $w(x) f(x) \to 0$ as $x \to x_j$*. The sufficiency of this condition immediately follows if we take $P_{r-1} \equiv 0$ as test polynomials in the definition (1.19). On the other hand, if $\omega_\varphi^r(f, t)_\omega^* \to 0$ as $t \to 0$, then it follows from the theorem that $E_n(f)_w \to 0$ as $n \to \infty$. Thus, for every k we can choose a polynomial P_{n_k} such that

$$\|w(f - P_{n_k})\| < \frac{1}{k}.$$

This shows first of all that f has to be continuous away from the nodes x_j. Furthermore, since $w(x) P_{n_k}(x) \to 0$ as $x \to x_j$, it follows that $w(x) |f(x)| \leqslant 1/k$ if x is sufficiently close to any of the x_j's, and this proves that $w(x) f(x) \to 0$ as $x \to x_j$.

2. PROOF OF THEOREMS 1.1 AND 1.3

We shall simultaneously prove Theorems 1.1 and 1.3. The proof follows well known paths.

We introduce the K-functional

$$K_r(f, t)_w = \inf_{g^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}} (\|w(f-g)\| + t^r \|w\varphi^r g^{(r)}\|). \tag{2.1}$$

Recall that $\varphi(x)=\sqrt{1-x^2}$. In [2, Theorem 2.1.1] it was proven that in the unweighted case the (unweighted) modulus of smoothness $\omega_\varphi^r(f, t)$ is equivalent to this K-functional in the sense that

$$\frac{1}{C}\,\omega_\varphi^r(f, t)\leqslant K_r(f, t)\leqslant C\omega_\varphi^r(f, t).$$

Now if we check that proof we can see that the same inequality with $0<t\leqslant 1/n$ holds true for w_n,

$$\frac{1}{C}\,\omega_\varphi^r(f, t)_{w_n}\leqslant K_r(f, t)_{w_n}\leqslant C\omega_\varphi^r(f, t)_{w_n}, \qquad 0<t\leqslant 1/n. \tag{2.2}$$

In fact, in all of the estimates in [2] the weight $w_n(x)$ can be inserted if we take into account that $|x-y|\leqslant M\Delta_n(x)$ implies

$$\frac{1}{C_M}\leqslant w_n(x)/w_n(y)\leqslant C_M \tag{2.3}$$

with a constant C_M depending only on M, which is an immediate consequence of (1.6).

Now we make use of the following Jackson–Favard type inequality that was proven in [4, Theorem 1.1]: *Let w be a doubling weight on $[-1, 1]$. Then for every positive integer r there is a constant C depending only on r and the doubling constant of w, such that for all g for which $g^{(r-1)}$ is locally absolutely continuous on $(-1, 1)$, we have*

$$E_n(g)_{w_n}\leqslant\frac{C}{n^r}\,\|w_n\varphi^r g^{(r)}\|. \tag{2.4}$$

On applying this to any g, making use of

$$E_n(f)_{w_n}\leqslant\|w_n(f-g)\|+E_n(g)_{w_n},$$

and then taking infimum for all possible g we arrive at

$$E_n(f)_{w_n}\leqslant CK_r(f, 1/n)_{w_n},$$

and then (1.7) follows from (2.2).

With this (1.11) has also been verified, for every A^* weight is also doubling. Next we prove (1.12). We use the following Bernstein-type inequality (see [3, (7.29)]): *If w is an A^* weight, then*

$$\|w\varphi^r P_n^{(r)}\|\leqslant Cn^r\,\|wP_n\| \tag{2.5}$$

for all polynomials P_n of degree at most n with a constant C independent of n and P_n that depends actually only on the A^* constant of w.

Note that together with w also every w_m is an A^* with A^* constant independent of m, therefore (2.5) can be applied with w replaced by w_m.

We have by (2.2)

$$\omega_\varphi^r(f, 1/n)_{w_n} \leqslant C\left(\|w_n(f-P_n^*)\| + \frac{1}{n^r}\|w_n\varphi^r(P_n^*)^{(r)}\|\right), \tag{2.6}$$

where P_k^* denotes the polynomial of degree at most k that approximates best f with the weight w_n (i.e., n in w_n is fixed, and the degree k of P_k^* varies). Here the first term on the right is $E_n(f)_{w_n}$. Let 2^m be the largest possible power of 2 not bigger than n. With this m we write

$$P_n^* = (P_n^* - P_{2^m}^*) + (P_{2^m}^* - P_{2^{m-1}}^*) + (P_{2^{m-1}}^* - P_{2^{m-1}}^*) + \cdots + (P_1^* - P_0^*) + P_0^*, \tag{2.7}$$

and then the second term in the previous inequality can be estimated via the Bernstein inequality (2.5) with w replaced by w_n:

$$\begin{aligned} Cn^{-r}\|w_n\varphi^r(P_n^*)^{(r)}\| \leqslant{}& Cn^{-r}n^r\|w_n(P_n^* - P_{2^m}^*)\| \\ &+ Cn^{-r}\sum_{k=1}^{m} 2^{kr}\|w_n(P_{2^k}^* - P_{2^{k-1}}^*)\| \\ &+ Cn^{-r}\|w_n(P_1^* - P_0^*)\|. \end{aligned} \tag{2.8}$$

Finally, if we subtract and add f, we can see that the norms in the sum on the right are at most $E_{2^k}(f)_{w_n} + E_{2^{k-1}}(f)_{w_n}$, and the terms before and after the sum can be similarly handled. Thus, in view of (2.6), we have obtained

$$\omega_\varphi^r(f, 1/n)_{w_n} \leqslant C\left(E_n(f)_{w_n} + \frac{1}{n^r}\sum_{k=0}^{m} 2^{kr}E_{2^k}(f)_{w_n} + \frac{1}{n^r}E_0(f)_{w_n}\right),$$

which can be written in the form

$$\omega_\varphi^r(f, 1/n)_{w_n} \leqslant \frac{C}{n^r}\sum_{k=0}^{m}(k+1)^{r-1}E_k(f)_{w_n}$$

using the monotonicity of $E_k(f)_{w_n}$ in k.

This is the inequality (1.13), and to prove (1.12) it remains to show that the terms $E_k(f)_{w_n}$ on the right can be replaced by $E_k(f)_{w_k}$. However, this follows from (2.3) and the A^* property of w, namely (2.3) and (1.10) imply that if $k \leqslant n$ then $w_n(x) \leqslant Cw_k(x)$ for some constant C, and hence $E_k(f)_{w_n} \leqslant CE_k(f)_{w_k}$.

With this the proof of Theorem 1.3 is complete.

Finally, we prove (1.8). We need the following Bernstein-type inequality for doubling weights.

LEMMA 2.1. *Let w be a doubling weight and r a natural number. There is a constant C such that for all natural numbers n and all polynomials P_n of degree at most n we have*

$$\|w_n \varphi^r P_n^{(r)}\| \leqslant Cn^r \, \|w_n P_n\|. \tag{2.9}$$

A warning is appropriate here: though this is the analogue of (2.5) for w_n, and though we have used in the proof above (2.5) for w_n (and for A^* weights), we shall not be able to derive (1.8) with $s=0$ as in (1.21). The problem is that (2.5) holds uniformly in w_n instead of w if w is an A^*, and in (2.9) the n in w_n and in P_n match each other, so e.g. even if k is much smaller than n we could only assert

$$\|w_n \varphi^r P_k^{(r)}\| \leqslant Cn^r \, \|w_n P_k\|,$$

and not the same inequality with Ck^r on the right (which follows from (2.5) when w is an A^* weight function). Therefore, even if we have (2.9) the above proof of (1.12) needs modification. We shall first point out these modifications, and then return to the proof of Lemma 2.1.

Let P_n^{**} denote the best polynomial approximant to f with weight w_n; i.e., now $P_{2^k}^{**}$ is taken with respect to the weight w_{2^k} and not with respect to w_n. Now follow the proof of (1.12) above everywhere replacing P_l^* by P_l^{**}. It is easy to see that for $l \leqslant n$ we have

$$\Delta_n(x) \leqslant \Delta_l(x) \leqslant \frac{n^2}{l^2} \Delta_n(x),$$

from which it immediately follows that

$$w_n(x) \leqslant \left(\frac{n}{l}\right)^2 w_l(x).$$

Furthermore, if $k \sim l$, then $w_k(x) \sim w_l(x)$, therefore $E_k(f)_{w_l} \sim E_k(f)_{w_k}$ if $k \sim l$. Thus, we obtain instead of (2.8)

$$\begin{aligned} \|w_n \varphi^r (P_n^{**})^{(r)}\| \leqslant {} & \|w_n \varphi^r (P_n^{**} - P_{2^m}^{**})^{(r)}\| \\ & + \sum_{k=1}^{m} \|w_n \varphi^r (P_{2^k}^{**} - P_{2^{k-1}}^{**})^{(r)}\| \\ & + \|w_n \varphi^r (P_1^{**} - P_0^{**})^{(r)}\| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \|w_n \varphi^r (P_n^{**} - P_{2^m}^{**})^{(r)}\| \\
&\quad + \sum_{k=1}^{m} \left(\frac{n}{2^k}\right)^2 \|w_{2^k} \varphi^r (P_{2^k}^{**} - P_{2^{k-1}}^{**})^{(r)}\| \\
&\quad + n^2 \|w_1 \varphi^r (P_1^{**} - P_0^{**})^{(r)}\|.
\end{aligned}
$$

Now here we can already apply the Bernstein inequality (2.9) to conclude

$$
\begin{aligned}
\|w_n \varphi^r (P_n^{**})^{(r)}\| &\leqslant Cn^r \|w_n (P_n^{**} - P_{2^m}^{**})\| \\
&\quad + C \sum_{k=1}^{m} \left(\frac{n}{2^k}\right)^2 2^{kr} \|w_{2^k} (P_{2^k}^{**} - P_{2^{k-1}}^{**})\| \\
&\quad + Cn^2 \|w_1 (P_1^{**} - P_0^{**})\|,
\end{aligned}
$$

and in a standard way this yields for $r > 2$ the estimate

$$
\omega_\varphi^r \left(f, \frac{1}{n}\right)_{w_n} \leqslant \frac{C}{n^{r-2}} \sum_{k=0}^{n} (k+1)^{r-3} E_k(f)_{w_k},
$$

which is just a different form of (1.8).

This proves Theorem 1.1 pending the proof of Lemma 2.1.

Proof of Lemma 2.1. First we verify the statement for $r = 1$.

The proof is based on the following lemma, for which see [3, (7.34)–(7.35)]:

LEMMA 2.2. *Let w be a doubling weight. Then there are polynomials R_n of degree at most n such that*

$$
\frac{1}{C} w_n(x) \leqslant R_n(x) \leqslant C w_n(x), \qquad x \in [-1, 1], \tag{2.10}
$$

and

$$
\varphi(x) \, |R_n'(x)| \leqslant Cn w_n(x), \qquad x \in [-1, 1] \tag{2.11}
$$

with a constant C depending only on the doubling constant of w.

Now the proof of (2.9) for $r = 1$ easily follows from (2.10) and (2.11). In fact, on applying these and Bernstein's inequality (2.5) for $w \equiv 1$ we can obtain from the formula $P_n' R_n = (P_n R_n)' - P_n R_n'$ that

$$
\begin{aligned}
w_n(x) \, \varphi(x) \, |P_n'(x)| &\leqslant C\varphi(x) \, |P_n'(x) \, R_n(x)| \\
&\leqslant C\varphi(x) \, |(P_n(x) \, R_n(x))'| + C\varphi(x) \, |P_n(x) \, R_n'(x)| \\
&\leqslant C2n \, \|P_n R_n\| + Cn \, |P_n(x) \, w_n(x)| \leqslant Cn \, \|w_n P_n\|.
\end{aligned}
$$

Finally, we show that the $r=1$ case implies the general one. To this end we need one more consequence of Lemma 2.2, namely that if w is a doubling weight, then with $I_n=[-1+1/n^2, 1-1/n^2]$ we have for all polynomials P_n of degree at most n the inequality

$$\|w_n P_n\| \leqslant C \|w_n P_n\|_{I_n}$$

with a constant C that depends only on the doubling constant of w. In fact, from the classical Remez inequality (cf. [3, Sect. 5.1])

$$\|P_n\| \leqslant C_M \|P_n\|_{I_{Mn}}$$

valid for every M with a constant C_M depending only on M we obtain with the polynomials R_n from Lemma 2.2 that

$$\|w_n P_n\| \leqslant C \|P_n R_n\| \leqslant C \|P_n R_n\|_{I_n} \leqslant C \|w_n P_n\|_{I_n}.$$

Consider now the weights $\varphi^j w_n$ with $0 \leqslant j \leqslant r$. It easily follows from (1.6) that these are doubling weights with doubling constants depending only on r and the doubling constant of w. Furthermore $(\varphi^j w_n)_n(x) \sim (\varphi(x) + 1/n)^j w_n(x)$ for $x \in [-1, 1]$, and hence $(\varphi^j w_n)_n(x) \sim \varphi(x)^j w_n(x)$ for $x \in I_n$, and otherwise $\varphi(x)^j w_n(x) \leqslant C(\varphi^j w_n)_n(x)$. Therefore, it follows from the already proven $r=1$ case of (2.9) that

$$\begin{aligned} \|w_n \varphi^r P_n^{(r)}\| &\leqslant C \|\varphi(w_n \varphi^{r-1})_n P_n^{(r)}\| \\ &\leqslant Cn \|(w_n \varphi^{r-1})_n P_n^{(r)}\| \leqslant Cn \|(w_n \varphi^{r-1})_n P_n^{(r-1)}\|_{I_n} \\ &\leqslant Cn \|w_n \varphi^{r-1} P_n^{(r-1)}\|. \end{aligned}$$

Now this is already in the form that can be iterated, and r-fold iteration gives (2.9), and with it the proof of Lemma 2.1, and also the proof of Theorem 1.1, is complete. ∎

3. PROOF OF THEOREM 1.4

3.1. *Proof of* (1.21)

We start with the proof of

$$\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f-P_{r-1})\|_{I_{j,1/n}} \leqslant \frac{C}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_w \tag{3.1}$$

for any $1 \leqslant j \leqslant N$. Let us first consider the case when $x_j \neq \pm 1$ and when γ_j is either not an integer, or it is an integer bigger than r.

Let P_k be the best polynomial approximant of degree at most k of f with weight w. We note first of all, that Bernstein's inequality (2.5) is true for w, and therefore exactly as in the proof of Theorem 1.3 we obtain

$$\|w\varphi^r P_n^{(r)}\| \leqslant C \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_w \tag{3.2}$$

(recall that $\varphi(x) = \sqrt{1-x^2}$).

It is clear that

$$\begin{aligned} &\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f-P_{r-1})\|_{I_{j,\,1/n}} \\ &\quad \leqslant \|w(f-P_n)\| + \inf_{\deg(P_{r-1}) \leqslant r-1} \|w(P_n - P_{r-1})\|_{I_{j,\,1/n}}. \end{aligned} \tag{3.3}$$

The first term on the right is $E_n(f)_w$, and for the second term we apply the following lemma (see [4, Lemma 2]):

Lemma A. *Let r be a positive integer, $a>0$, and $v(t) = |t|^\gamma$, $\gamma > 0$, where either $\gamma > r$, or γ is not an integer. Then for any function g for which $g^{(r-1)}$ is locally absolutely continuous on $[-a, a]\backslash\{0\}$ there are polynomials P_{r-1} of degree at most $r-1$ such that for $x \in [-1/n, 1/n]$, $n \geqslant 1/a$*

$$v(x)\,|g(x) - P_{r-1}(x)| \leqslant \frac{C_{a,r,\gamma}}{n^r} (\|vg^{(r)}\|_{[-a,a]} + \|vg\|_{[-a,a]}).$$

We apply this with the origin replaced by x_j to $g = P_n$, $\gamma = \gamma_j$ and to some small a such that in the interval $[x_j - a, x_j + a]$ there are no further x_i's. It follows that

$$\begin{aligned} &\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(P_n - P_{r-1})\|_{I_{j,\,1/n}} \\ &\quad \leqslant \frac{C}{n^r} (\|wP_n^{(r)}\|_{[x_j-a,\,x_j+a]} + \|wP_n\|_{[x_j-a,\,x_j+a]}) \\ &\quad \leqslant \frac{C}{n^r} (\|w\varphi^r P_n^{(r)}\|_{[x_j-a,\,x_j a]} + \|wP_n\|_{[x_j-a,\,x_j+a]}), \end{aligned}$$

where, in the last inequality, we used the fact that φ is strictly positive on $[x_j - a, x_j + a]$. Here the first term on the right is estimated by (3.2). As for the second term, we have

$$\|wP_n\|_{[x_j-a,\,x_j+a]} \leqslant \|wf\|_{[x_j-a,\,x_j+a]} + \|w(f-P_n)\|. \tag{3.4}$$

Collecting the estimates so far we can see that if A denotes the right hand side of (3.1), then

$$\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f-P_{r-1})\|_{I_{j,\,1/n}} \leqslant C\left(A+\frac{1}{n^r}\|wf\|\right). \tag{3.5}$$

Now apply this to $f-P_0$ instead of f. Notice that the left hand side does not change, and nor does A, so we obtain

$$\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f-P_{r-1})\|_{I_{j,\,1/n}} \leqslant C\left(A+\frac{1}{n^r}\|w(f-P_0)\|\right). \tag{3.6}$$

Since here the second term can be incorporated into A, we finally arrive at (3.1).

Next we consider the case when still $x_j \neq \pm 1$, but γ_j is an integer not bigger than r. We factor out $q(x)=|x-x_j|^{\gamma_j}$ from the weight, and set $W(x) = w(x)/q(x)$. We use again (3.3) and notice that in the second term on the right hand side we can replace $w(x)$ by $q(x)$. Now consider the function $q(x)\,P_n(x)$. This vanishes at x_j together with its first γ_j-1 derivatives, so its Taylor polynomial of degree $r-1$ about x_j contains the factor $(x-x_j)^{\gamma_j}$, and hence this Taylor polynomial is of the form $(x-x_j)^{\gamma_j}\,P_{r-1}(x)$ with a polynomial of degree at most $r-1$ (actually smaller than $r-\gamma_j$). Thus, for the second term on the right of (3.3) we obtain from the remainder formula for Taylor polynomials

$$\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(P_n-P_{r-1})\|_{I_{j,\,1/n}} \leqslant \frac{1}{n^r}\|(qP_n)^{(r)}\|_{I_{j,\,1/n}}.$$

Here we can insert $W\varphi^r$ into the right hand side, for this factor is strictly positive on the interval $I_{j,\,1/n}$. Now we apply the decomposition (2.7) in the form

$$\begin{aligned} qP_n &= (qP_n - qP_{2^m}) + (qP_{2^m} - qP_{2^{m-1}}) + (qP_{2^{m-1}} - qP_{2^{m-2}}) \\ &\quad + \cdots + (qP_1 - qP_0) + qP_0, \end{aligned}$$

and use Bernstein's inequality with weight W as in (2.8) to obtain

$$\begin{aligned} Cn^{-r}\,\|W\varphi^r(qP_{2^m})^{(r)}\| &\leqslant Cn^{-r}n^r\,\|W(qP_n-qP_{2^m})\| \\ &\quad + Cn^{-r}\sum_{k=0}^{m} 2^{kr}\,\|W(qP_{2^k}-qP_{2^{k-1}})\| \\ &\quad + Cn^{-r}\,\|WqP_0\|. \end{aligned}$$

Since here $Wq \equiv w$, we obtain by the procedure applied after (2.8) that

$$\inf_{\deg(P_{r-1}) \leqslant r-1} \|w(f-P_{r-1})\|_{I_{j,\,1/n}} \leqslant C\left(A+\frac{1}{n^r}\|wP_0\|\right).$$

Finally, here the second term can be handled as in (3.4)–(3.6), and the proof of (3.1) is complete in this case, as well.

Finally we consider (3.1) when x_j is one of the endpoints, say $x_j = -1$. We use again (3.3), but now for estimating the second term on the right hand side we use the following lemma (see [4, Lemma 3]) instead of Lemma A:

LEMMA B. *Let r be a positive integer, $a>0$, $v(t)=t^{\gamma}$, $\gamma>0$ and $V(t)=t^{\gamma+r/2}$. Then for any function g for which $g^{(r-1)}$ is locally absolutely continuous on $(0, a]$ and for every $n \geqslant 1/a$ there are polynomials P_{r-1} of degree at most $r-1$ such that for $x \in [0, 1/n^2]$*

$$v(x)\,|g(x)-P_{r-1}(x)| \leqslant \frac{C}{n^r}(\|Vg^{(r)}\|_{[0,\,a]}+\|vg\|_{[0,\,a]}), \tag{3.7}$$

where C depends only on γ and r.

Note that in this lemma there is no restriction on γ, so in this case we can have a unified proof of (3.1) irrespectively of the value of γ_j. We apply this lemma with the origin replaced by -1 to the function P_n with $\gamma=\gamma_j$. Since $|x+1|^{\gamma_j} \sim w(x)$ and $|x+1|^{\gamma_j+r/2} \sim w(x)\,\varphi^r(x)$, we can conclude (recall that in the present case the $I_{j,\,1/n}$ is the interval $[-1, -1, +1/n^2]$)

$$\begin{aligned}\inf_{\deg(P_{r-1}) \leqslant r-1} &\|w(P_n-P_{r-1})\|_{I_{j,\,1/n}}\\ &\leqslant \frac{C}{n^r}(\|w\varphi^r P_n^{(r)}\|_{[-1,\,-1+a]}+\|wP_n\|_{[-1,\,-1+a]}),\end{aligned}$$

and from here the proof is identical to the one used before.

With this the proof of (3.1) is complete.

Next we prove that for each $1 \leqslant j \leqslant N-1$ we have for $|h| \leqslant 1/n$

$$\|w(x)\,\Delta^r_{h\varphi(x)}f(x)\|_{J_{j,\,h}} \leqslant \frac{C}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_w. \tag{3.8}$$

On taking here the supremum for all $|h| \leqslant 1/n$, and adding these estimates together for all j we obtain that the main part modulus (i.e., the first sum in $\omega^r_{\varphi}(f, 1/n)^*_w$, cf. (1.19)) is bounded by the right hand side of (3.8), and this together with (3.1) yields (1.21).

In forming $\Delta^r_{h\varphi(x)} f(x)$ we take a linear combination of the function values $f(x+(r/2-s)\, h\varphi(x))$, $0 \leqslant s \leqslant r$. However, the norm of $\Delta^r_{h\varphi(x)} f(x)$ is taken only for values of x for which $[x-(r/2)\, h\varphi(x), x+(r/2)\, h\varphi(x)] \subset J_{j,h}$, and the definition of $J_{j,h}$ shows that in this case for $t \in [x-(r/2)\, h\varphi(x), x+(r/2)\, h\varphi(x)]$ we have

$$w(t) \sim w(x) \qquad \text{and} \qquad \varphi(t) \sim \varphi(x) \tag{3.9}$$

uniformly in h and x. Therefore, we can freely exchange a value of w or φ for $t \in [x-(r/2)\, h\varphi(x), x+(r/2)\, h\varphi(x)]$ for another value in this same interval.

Consider now the inequality

$$\|w\Delta^r_{h\varphi} f\|_{J_{j,h}} \leqslant \|w\Delta^r_{h\varphi}(f-P_n)\|_{J_{j,h}} + \|w\Delta^r_{h\varphi} P_n\|_{J_{j,h}}, \tag{3.10}$$

where P_n again denotes the best polynomial approximant of f of degree at most n with weight w. As a first application of (3.9) we obtain that for $[x-(r/2)\, h\varphi(x), x+(r/2)\, h\varphi(x)] \subset J_{j,h}$ we have

$$\begin{aligned} w(x)\, &|\Delta^r_{h\varphi(x)}(f-P_n)(x)| \\ &\leqslant \sum_{s=0}^{r} \binom{r}{s} w(x) \left|(f-P_n)\left(x+\left(\frac{r}{2}-s\right) h\varphi(x)\right)\right| \\ &\leqslant \sum_{s=0}^{r} \binom{r}{s} Cw\left(x+\left(\frac{r}{2}-s\right) h\varphi(x)\right) \left|(f-P_n)\left(x+\left(\frac{r}{2}-s\right) h\varphi(x)\right)\right| \\ &\leqslant C2^r E_n(f)_w. \end{aligned} \tag{3.11}$$

To estimate the second term in (3.10) we use (cf. [2, (2.4.5)]) that $\Delta^r_{h\varphi(x)} P_n(x)$ equals

$$\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} P_n^{(r)}(x+u_1+\cdots+u_r)\, du_1 \cdots du_r,$$

which in absolute value is at most

$$Ch^r(\varphi(x))^r\, \|P_n^{(r)}\|_{[x-rh\varphi(x)/2,\, x+rh\varphi(x)/2]}.$$

Thus,

$$|\Delta^r_{h\varphi(x)} P_n(x)| \leqslant Ch^r(\varphi(x))^r\, \|P_n^{(r)}\|_{[x-rh\varphi(x)/2,\, x+rh\varphi(x)/2]}.$$

On multiplying this with $w(x)$ and applying (3.9) we obtain

$$\|w\Delta^r_{h\varphi} P_n\|_{J_{j,h}} \leqslant Ch^r\, \|w\varphi^r P_n^{(r)}\|,$$

and for the right hand side we can apply (3.2). Taking into account (3.10) and (3.11) we can see that (3.8) is true, and the proof of (1.21) is complete.

3.2. *Proof of* (1.20)

We shall deduce (1.20) from Theorem 1.3.

Let $\psi(x)$ be an infinitely differentiable function that is zero for negative x, equals 1 for $x > 1$ and otherwise $0 \leqslant \psi(x) \leqslant 1$, and let $P_{r-1,\,j}$ be the best polynomial approximant of degree at most $r-1$ of f with weight w on the interval $I_{j,\,1/n}$. We set (for a verbal description of the following expression see the discussion below)

$$\begin{aligned} g(x) = {} & P_{r-1,\,1}(x)\left(1-\psi\left(2n^2\left(x-\left(-1+\frac{1}{4n^2}\right)\right)\right)\right) \\ & + f(x)\,\psi\left(2n^2\left(x-\left(-1+\frac{1}{4n^2}\right)\right)\right)\left(1-\psi\left(2n\left(x-\left(x_2-\frac{1}{n}\right)\right)\right)\right) \\ & + P_{r-1,\,N}\psi\left(2n^2\left(x-\left(1-\frac{3}{4n^2}\right)\right)\right) \\ & + f(x)\,\psi\left(x-\left(x_{N-1}+\frac{1}{2n}\right)\right)\left(1-\psi\left(2n^2\left(x-\left(1-\frac{3}{4n^2}\right)\right)\right)\right) \\ & + \sum_{j=2}^{N-1} P_{r-1}(x)\,\psi\left(2n\left(x-\left(x_j-\frac{1}{n}\right)\right)\right)\left(1-\psi\left(2n\left(x-\left(x_j+\frac{1}{2n}\right)\right)\right)\right) \\ & + \sum_{j=2}^{N-2} f(x)\,\psi\left(2n\left(x-\left(x_j+\frac{1}{2n}\right)\right)\right)\left(1-\psi\left(2n\left(x-\left(x_{j+1}-\frac{1}{n}\right)\right)\right)\right). \end{aligned}$$

This function coincides with f outside $\bigcup_{j=1}^{N} I_{j,\,1/n}$, coincides with $P_{r-1,\,j}$ on the interval $I_{j,\,1/2n}$, and on the intervals $[x_j - 1/n,\ x_j - 1/2n]$ and $[x_j + 1/2n,\ x_j + 1/n]$ it is a convex combination of these two functions (with obvious modification if $x_j = \pm 1$). Thus, $f-g$ is zero on $\bigcup_{j=1}^{N_1} J_{j,\,1/n}$, and on the interval $I_{j,\,1/n}$ we have $|f-g| = |\chi_j^*(f-P_{r-1,\,j})|$ with a χ_j^* lying between 0 and 1, and so by the definition of the modulus of continuity $\omega_\varphi^r(f;\tau)_w^*$ we have

$$\|w(f-g)\| \leqslant \omega_\varphi^r(f;1/n)_w^*. \tag{3.12}$$

Let now $|h| \leqslant 1/10rn$. We have to estimate $\Delta_{h\varphi(x)}^r g(x)$ in three different ranges.

- Since g equals a polynomial of degree at most $r-1$ on each of the intervals $I_{j,\,1/2n}$, it follows that $\Delta_{h\varphi(x)}^r g(x)$ is zero if the interval $[x - rh\varphi(x)/2,\ x + rh\varphi(x)/2]$ does not belong to any of the intervals $J_{j,\,h}$ (which means that it must belong to some $I_{j,\,1/2n}$)

- If $[x-rh\varphi(x)/2, x+rh\varphi(x)/2]$ belongs to any of the $J_{j,1/n}$, then $\Delta^r_{h\varphi(x)}g(x)$ equals $\Delta^r_{h\varphi(x)}f(x)$.
- Finally, in the remaining case there is a j such that

$$[x-rh\varphi(x)/2, x+rh\varphi(x)/2] \cap [x_j-2/n, x_j-1/2n] \neq \emptyset$$

or

$$[x-rh\varphi(x)/2, x+rh\varphi(x)/2] \cap [x_j+1/2n, x_j+2/n] \neq \emptyset$$

(with obvious modification when $x_j = \pm 1$). Consider for example, the latter case. On the interval $[x_j+1/4n, x_j+3/n]$, that contains $[x-rh\varphi(x)/2, x+rh\varphi(x)/2]$, we can apply (3.9) to conclude that

$$\begin{aligned} w(x)\,|\Delta^r_{h\varphi(x)}g(x)| &\leqslant w(x)\,|\Delta^r_{h\varphi(x)}f(x)| + w(x)\,|\Delta^r_{h\varphi(x)}(f-g)(x)| \\ &\leqslant w(x)\,|\Delta^r_{h\varphi(x)}f(x)| + C\,\|w(f-g)\|_{[x_j+1/4n,\, x_j+3/n]}. \end{aligned}$$

Note that in the last two ranges we have $w_n(x) \sim w(x)$, and in the first range, when this is not satisfied, the r-th difference $\Delta^r_{h\varphi(x)}g(x)$ is actually zero. Therefore, from the consideration above we can deduce that

$$\|w_n(x)\,\Delta^r_{h\varphi(x)}g(x)\| \leqslant \sum_{j=1}^{N} \|w(x)\,\Delta^r_{h\varphi(x)}f(x)\|_{J_{j,h}} + C\,\|w(f-g)\|.$$

Since this is true for all $|h| \leqslant 1/10rn$, we obtain from (3.12) and the definition of our moduli of smoothness that

$$\omega^r_\varphi\left(g, \frac{1}{10rn}\right)_{w_n} \leqslant C\omega^r_\varphi\left(f; \frac{1}{n}\right)^*_w.$$

Using this and (2.2) twice it follows that

$$\begin{aligned} \omega^r_\varphi\left(g, \frac{1}{n}\right)_{w_n} &\leqslant CK_r\left(g, \frac{1}{n}\right) \leqslant C(10r)^r\, K_r\left(g, \frac{1}{10rn}\right)_{w_n} \\ &\leqslant C_1\omega^r_\varphi\left(g, \frac{1}{10rn}\right)_{w_n} \leqslant C_2\omega^r_\varphi\left(f, \frac{1}{n}\right)^*_w, \end{aligned}$$

where the second inequality is an immediate consequence of the definition of the K-functional (see (2.1)).

Now this inequality, (3.12) and Theorem 1.3 yield

$$
\begin{aligned}
E_n(f)_w &\leqslant \|w(f-g)\| + E_n(g)_w \leqslant \|w(f-g)\| + CE_n(g)_{w_n} \\
&\leqslant \|w(f-g)\| + C\omega_\varphi^r\left(g, \frac{1}{n}\right)_{w_n} \\
&\leqslant C\omega_\varphi^r\left(f, \frac{1}{n}\right)_w^*,
\end{aligned}
$$

where, at the second inequality we also used that $w(x) \leqslant Cw_n(x)$ for some C. This completes the proof.

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